JOURNAL OF ALGEBRA **109**, 468–478 (1987)

# Essential Prime Divisors and Projectively Equivalent Ideals

D. KATZ

*Department of Mathematics, University of Kansas Lawrence, Kansas 66045*

S. MCADAM*

*Department of Mathematics, University of Texas Austin, Texas 78712*

J. S. OKON*

*Department of Mathematics, California State University,*
*San Bernadino, California 92407*

AND

L. J. RATLIFF, JR.*

*Department of Mathematics, University of California, Riverside, Califonia 92521*

*Communicated by Melvin Hochster*

Received July 31, 1985

## INTRODUCTION

Let $I$ be an ideal in a Noetherian ring $R$. We concern ourselves with the essential prime divisors of $I$, an interesting subset of Ass $R/I^n$, for all large $n$. We first take $I = bR$ with $b$ a regular element of $R$. We show that there is a ring $T$, with $R \subseteq T \subseteq R_b$, such that $T$ is a finite $R$-module and the essential primes of $bT$ are exactly the prime divisors of $bT$. We next consider an arbitrary ideal $I$, and apply our principal arguments to the element $u$ in the Rees ring of $I$. We thereby deduce that there is an ideal $J$ projectively equivalent to $I$, such that the set of essential primes of $I$ equals the set $\bigcup$ Ass $R/J^n$, over $n = 1, 2, 3, \ldots$.

*Notation.* Let $I$ be an ideal in a Noetherian ring $R$. We will use $\mathscr{R}$ (or

---

* These authors were partially supported by grants from the National Science Foundation.

0021-8693/87 $3.00

$R(I)$) to be the Rees ring of $I$. Thus $\mathcal{R} = R[u, It]$. Here, $t$ is an indeterminate, and $u = t^{-1}$. If $R$ is local, $R^*$ will denote its completion.

$$A^*(I) = \operatorname{Ass} R/I^n \quad \text{for sufficiently large } n,$$

(the persistent primes of $I$).

$$Q(I) = \{P \in \operatorname{Spec} R \mid I \subseteq P \text{ and there is a } z \in \operatorname{Ass} R_P^* \text{ with } P_P^* \\ \text{minimal over } IR_P^* + z\}$$

(the quintessential primes of $I$).

$$E(I) = \{P \cap R \mid P \in Q(u\mathcal{R})\},$$

(the essential primes of $I$).

$$\mathscr{E}(R) = \{P \in \operatorname{Spec} R \mid P \notin \operatorname{Ass} R \text{ and } R_P^* \text{ has a depth 1 prime} \\ \text{divisor of } 0\},$$

(the essential primes of $R$).

*A Remark about Notation and Terminology.* The notation and names of $E(I)$ and $Q(I)$ as given above, represent a change from usage in earlier publications, in particular from the references quoted herein. We attach an appendix which offers our reasons for making these changes, and which will be helpful for translating results in the references into the new terminology.

PROPOSITION 1. (a) *If $S$ is a multiplicatively closed subset of $R$ and $P$ is a prime disjoint from $S$, then $P \in E(I)$ if and only if $P_S \in E(I_S)$.*

(b) *If $R \subseteq T$ is a faithfully flat extension of Noetherian rings, then $P \in E(I)$ if and only if there is a $Q \in E(IT)$ with $Q \cap R = P$.*

(c) *Let the ring $T$ be a finite module extension of $R$. If $P \in E(I)$, then there is a $Q \in E(IT)$ with $Q \cap R = P$. If also $z \in \operatorname{Ass} T$ implies $z \cap R \in \operatorname{Ass} R$, then the converse holds as well.*

(d) *If $b$ is a regular element of $R$, then $E(bR) = Q(bR)$.*

(e) *Let $P \in \mathscr{E}(R)$, and let $b$ be an element in $P$ whose image in $R_P$ is regular. Then $P \in E(bR)$.*

(f) *If $b$ is a regular element of $R$ and $P \in E(bR)$, then $P$ is a prime divisor of $bR$.*

(g) *If $I$ and $J$ are projectively equivalent ideals, then $E(I) = E(J)$.*

(h) *$E(I) \subseteq A^*(I)$, and these sets are finite.*

*Proof.* (a), (b), (c), and (d) are proved in [2, (2.5.1), (2.5.3), (2.5.4), and (2.5.8)]. (e) is straightforward from part (d) and the definitions. For (f), if $P \in E(bR)$, then [2, (2.3.3)] shows that $P$ is a prime divisor of $b^n R$ for some $n \geqslant 0$. Since $b$ is regular, $P$ is prime divisor of $bR$. (g) is proved in [2, (2.5.6)]. The containment in (h) is given in [2, (2.3.3)]. Finally, $A^*(I)$ is well defined, and finite by [3, Corollary 1.5].

We need a powerful result about ideal transforms.

DEFINITION. Let $I$ be a regular ideal in a Noetherian ring $R$. The ideal transform $T(I) = \{ y \mid y$ is in the total quotient ring of $R$, and for some $n \geqslant 0$, $yI^n \subseteq R\}$.

PROPOSITION 2. *Let $I$ be a regular ideal in a Noetherian ring $R$. Then $T(I)$ is a finite $R$-module if and only if $I \nsubseteq P$ for all $P \in \mathscr{E}(R)$.*

*Proof.* This follows from [3, Propositions 10.9 and 10.11].

We need an easy idea which is not easily expressed. The following definition corrects that situation.

DEFINITION. Suppose $K \subseteq H$ are rings and $x$ is an element of $K$ such that $K_X = H_X$. Let $U$ be a subset of Spec $K$, and suppose that $x \notin P$ for all $P \in U$. Let $W = \{P_x \cap H \mid P \in U\}$. Then we shall show that $x$ lifts $U$ to $W$. (In this case, there is a natural one-to-one inclusion preserving correspondence between $U$ and $W$, corresponding primes having the same height).

LEMMA 3. *Let $K$ be a Noetherian ring and let $b$ be a regular element of $K$. Let $E(bK) = \{Q_1, ..., Q_m\}$, and let $P_1, ..., P_n$ be the prime divisors of $bK$ which are not contained in $Q_1 \cup \cdots \cup Q_m$. Let $x$ be a regular element in $(P_1 \cap \cdots \cap P_n) - (Q_1 \cup \cdots \cup Q_m)$. Let $H = K_x \cap K_b$. Then*

(i) *$H$ is a finite $K$-module.*

(ii) *$K_x = H_x$.*

(iii) *$x$ lifts $\{P \in \text{Ass } K/bK \mid P$ is contained in some prime in $E(bK)\}$ to Ass $H/bH$. Also, no prime in Ass $H/bH$ contains $x$.*

(iv) *$x$ lifts $E(bK)$ to $E(bH)$.*

(v) *the maximal members of Ass $H/bH$ are identical to the maximal members of $E(bH)$.*

(vi) *If $K$ satisfies $\mathscr{R} = R(I) \subseteq K \subseteq R[u, t]$ with $K$ graded, and if $b = u$, then $x$ can be chosen to be homogeneous, and $H$ will be a graded ring with $\mathscr{R} \subseteq K \subseteq H \subseteq R[u, t]$.*

*Proof.* We first mention that we can always find an $x$ as in the statement. Since $b$ is regular, we see that $P_1 \cap \cdots \cap P_n \nsubseteq \bigcup \{Q \in \operatorname{Ass} K\} \cup (Q_1 \cup \cdots \cup Q_m)$, and so we use the prime avoidance lemma.

(i) One easily sees that $H$ is the ideal transform $T((b, x) K)$. By Proposition 1(e), any prime in $\mathscr{E}(K)$ which contains $b$ is automatically in $E(bK)$. Therefore, by the choice of $x$, no prime in $\mathscr{E}(K)$ contains $(b, x) K$. Thus by Propositin 2, $H$ is a finite $K$-module.

(ii) This is trivial.

(iii) Since $b$ is a unit in $K_b$, $bH = b(K_x \cap K_b) = bK_x \cap K_b = bK_x \cap K_x \cap K_b = bK_x \cap H = bH_x \cap H$. By standard facts about primary decomposition, we see that $x$ lifts $\{P \in \operatorname{Ass} K/bK \mid x \notin P\}$ to $\operatorname{Ass} H/xH$, so that the last statement in (iii) is true. Also, the choice of $x$ shows that $\{P \in \operatorname{Ass} K/bK \mid P$ is contained in some prime in $E(bK)\} = \{P \in \operatorname{Ass} K/bK \mid x \notin P\}$, so that the first statement is true.

(iv) By Proposition 1(f), primes in $E(bH)$ are always in $\operatorname{Ass} H/bH$, and so do not contain $x$, by (iii). Also, primes in $E(bK)$ do not contain $x$, by construction. Therefore, since $K_x = H_x$, it follows trivially from Proposition 1(a) that $x$ lifts $E(bK)$ to $E(bH)$.

(v) This follows easily from (iii), (iv), and Proposition 1(f).

(vi) Since $u$ is homogeneous, the primes $Q_1, ..., Q_m$, are all homogeneous, as are the primes in $\operatorname{Ass} K$. An easy variation of the standard prime avoidance lemma allows us to pick our $x$ to be homogeneous. Obviously $K \subseteq H \subseteq K_u = R[u, t]$. Since $y \in H$ exactly when $y \in R[u, t]$ and some positive power of the homogeneous element $x$ sends $y$ into $K$, we easily see that $H$ is a graded ring.

Recall that a local ring $(R, M)$ is unmixed if for every $z \in \operatorname{Ass} R^*$, depth $z =$ height $M$. (Thus, a complete local ring with a single prime divisor of zero is unmixed.) It is known that if $R$ is a Noetherian ring and $R_M$ is unmixed for all maximal ideals $M$, then $R$ is locally unmixed, i.e., $R_P$ is unmixed for all primes $P$. Also, if $I$ is an ideal in a locally unmixed Noetherian ring $R$, and if $R(I)$ is the Rees ring of $I$, then $R(I)$ is locally unmixed. See [5].

LEMMA 4. *Let $A \subseteq B \subseteq C$ be Noetherian rings with $A \subseteq B$ a faithfuly flat extension, and $B \subseteq C$ a finite module extension such that for all $z \in \operatorname{Ass} C$, $z \cap B \in \operatorname{Ass} B$. Suppose also that $C$ is locally unmixed. Let $b$ be a regular element of $A$ (so $b$ is still regular in $C$). Pick $x$ and $H = C_x \cap C_b$ as in Lemma 3 applied to $K = C$ and $b \in C$. Let $D = H \cap A_b$. Then $D$ is a finite $A$-module contained in $A_b$, and* $\operatorname{Ass} D/bD = E(bD)$.

*Proof.* Note that $D = C_x \cap A_b$. Since $b$ is a unit in both $C_b$ and $A_b$,

$bH = bC_x \cap C_b$ and $bD = bC_x \cap A_b$. Thus it is easy to verify that $bH \cap D = bD$. Therefore, primes in Ass $D/bD$ lift to primes in Ass $H/bH$.

If $Q \in E(bC)$, then by Proposition 1(d), $Q \in Q(bC)$, and so $Q_Q^*$ is minimal over $bC_Q^* + z$ for some $z \in \operatorname{Ass} C_Q^*$. Therefore depth $z = 1$. Since $C_Q$ is unmixed, height $Q =$ depth $z = 1$. Thus every prime in $E(bC)$ has height 1. Lemma 3(iv) now shows that all of the primes in $E(bH)$ have height 1. By Lemma 3(v) and Proposition 1(f), we see that Ass $H/bH = E(bH)$.

By Lemma 3(i), $H$ is a finite $C$-module, and by assumption $C$ is a finite $B$-module. Thus $H$ is a finite $B$-module. Since $B \subseteq B[D] \subseteq H$, $H$ is a finite $B[D]$-module. Also, primes in Ass $H$ contract to primes in Ass $B[D]$, since this holds between $B$ and $C$, and we are working in the total quotient rings of these two rings. By Proposition 1(c), primes in $E(bH)$ contract to primes in $E(bB[D])$. Also, $B[D] = B \otimes_A D$, so by Proposition 1(b), primes in $E(bB[D])$ contract to primes in $E(bD)$. Thus primes in $E(bH)$ contract to primes in $E(bD)$.

Combining the conclusions of the previous three paragraphs shows that Ass $D/bD = E(bD)$ (since one inclusion is by Proposition 1(f)). Also, since $H$ is a finite $B$-module, we see that $B[D]$ is a finite $B$-module. Since $B[D] = B \otimes_A D$, faithful flatness shows that $D$ is a finite $A$-module. Obviously $D \subseteq A_b$.

Let $R$ be a Noetherian ring with integral closure $R'$. Let $b$ be a regular element of $R$. If $T$ is a ring with $R \subseteq T \subseteq R'$ and $T$ a finite $R$-module, then Proposition 1(c) and (f) show that any prime in $E(bR)$ lifts to a prime divisor of $bT$. In general, the converse fails. However, our first main theorem shows that there exists such a $T$ for which the converse holds.

THEOREM 5. *Let $b$ be a regular element of the Noetherian ring $R$. Then there is a ring $T$ with $R \subseteq T \subseteq R_b$ such that $T$ is a finite $R$-module and* Ass *$T/bT = E(bT)$. Also, $P \in E(bR)$ if and only if $P$ lifts to a prime divisor of $bT$.*

*Proof.* Let $S = R - \bigcup \{P \in E(bR)\}$, and let $A = R_S$. As $A$ is semi-local (Proposition 1(h)), let $B$ equal the completion $A^*$. Let $q_1 \cap \cdots \cap q_n$ be a primary decomposition of 0 in $B$, and let $C = B/q_1 \oplus \cdots \oplus B/q_n$. There is a natural embedding of $B$ into $C$. Under it, we see that $b$, $A$, $B$, and $C$ satisfy the hypotheses of Lemma 4 (since every maximal localization of $C$ is a complete local ring with a single prime divisor of zero, and hence is unmixed). Let $D$ be as defined in that lemma. Then $R_S \subseteq D \subseteq (R_S)_b = (R_b)_S$. Also, if $R'$ is the integral closure of $R$, then since $D$ is a finite $R_S$-module, $D \subseteq R'_S$. Thus $R_S \subseteq D \subseteq (R_b \cap R')_S$. It is easy to find a finitely generated ring $F$ with $R \subseteq F \subseteq R_b \cap R'$, such that $F_S = D$. Obviously, $F$ is a finite $R$-module.

We now claim that $\{P \in \operatorname{Ass} F/bF \mid P$ is contained in a prime in

$E(bF)\} = E(bF)$. Suppose $P$ is in the first set, and that $P \subseteq Q \in E(bF)$. By Proposition 1(c), $Q \cap R \in E(bR)$, and so is disjoint from $S$. Thus $P \cap S = \emptyset$. Therefore, $P_S$ is a prime divisor of $bF_S = bD$. But $\operatorname{Ass} D/bD = E(bD)$, so $P_S \in E(bD) = E(bF_S)$, and so $P \in E(bF)$, by Proposition 1(a). This shows one containment of our claim. The other is by Proposition 1(f).

We now apply the construction of Lemma 3 to $K = F$ and $b \in F$. We let $T$ be the ring $H$ given by that lemma. Then $T$ is a finite $F$-module, hence a finite $R$-module, and $T \subseteq F_b = R_b$. Also by Lemma 3(iii) and (iv), and the claim we have just proved, we have $\operatorname{Ass} T/bT = E(bT)$. Thus the first conclusion of our theorem is proved. For the second, if $P \in E(bR)$ then $P$ lifts to a prime divisor of $bT$ by Proposition 1(c) and (f). Conversely, if $p$ is a prime divisor of $bT$, then $p \in E(bT)$, and so $p \cap R \in (bT)$ by Proposition 1(c).

The next corollary is easy, but it points out an important difference between arbitrary prime divisors and essential primes of a regular element.

COROLLARY 6. *Let $b$ be a regular element of a Noetherian ring $R$ having integral closure $R'$. Then $P \in E(bR)$ if and only if $P$ lifts to a prime divisor of $bT$ for every finitely generated ring $T$ with $R \subseteq T \subseteq R'$.*

*Proof.* This is easy by Proposition 1(c) and (f), and Theorem 5.

*Notation.* If $R$ is a Noetherian ring, we use $\mathscr{P}(R)$ to denote $\{P \in \operatorname{Spec} R \mid P_P \text{ has grade } 1\}$. Also, let $\mathscr{N}(R) = \mathscr{P}(R) - \mathscr{E}(R)$. (Note: It follows from Proposition 1, parts (a), (e), and (f), that $\mathscr{E}(R) \subseteq \mathscr{P}(R)$.)

In [4], a study is made of when $\mathscr{N}(R)$ is finite. For instance, if $R$ is semilocal, then $\mathscr{N}(R)$ is finite. Part (b) of the next corollary answers a question asked in [6, (6.7.2)].

COROLLARY 7. *Let $R$ be a Noetherian ring.*

(a) *Let $b$ be a nonnilpotent element of $R$, and let $K$ be the kernel of the canonical map $R \to R_b$. There is a ring $T$ with $R/K \subseteq T \subseteq R_b$ such that $T$ is a finite $R$-module, and* $\operatorname{Ass} T/bT = E(bT)$.

(b) *If $\mathscr{N}(R)$ is finite, then $b$ and $T$ can be chosen as in part* (a) *such that $\mathscr{N}(T) = \emptyset$.*

*Proof.* (a) We easily see that $b + K$ is regular in $R/K$. Also, $(R/K)_{b+K} = R_b$. Thus (a) follows from Theorem 5. For (b), since minimal primes cannot be in $\mathscr{P}(R)$, if $\mathscr{N}(R)$ is finite, we can find a nonnilpotent $b$ contained in the intersection of the primes in $\mathscr{N}(R)$. Pick $T$ as in part (a). If $P \in \mathscr{P}(T)$, we must show $P \in \mathscr{E}(T)$. If $bT \subseteq P$, then $P \in \operatorname{Ass} T/bT = E(bT) \subseteq \mathscr{E}(T)$. Thus suppose $bT \nsubseteq P$. Since $T_b = R_b$, we see that $P_b \in \mathscr{P}(T_b) = \mathscr{P}(R_b)$. Thus if $Q$ is the inverse image of $P_b$ in $R$, then

$Q \in \mathcal{P}(R)$. Also, $b \notin Q$. By choice of $b$, we have $Q \in \mathcal{E}(R)$. Thus $P_b = Q_b \in \mathcal{E}(R_b) = \mathcal{E}(T_b)$, and so $P \in \mathcal{E}(T)$.

We now begin considering an arbitrary ideal $I$ in a Noetherian ring $R$. Recall that $\mathcal{R}$ (or $R(I)$) will denote the Rees ring of $I$. We will apply the preceding ideas to $E(u\mathcal{R}) = Q(u\mathcal{R})$ (Proposition 1(d)), and use them to deduce information concerning $E(I) = \{P \cap R \mid P \in Q(u\mathcal{R})\}$.

THEOREM 8. *Let $I$ be an ideal in Noetherian ring $R$, and let $\mathcal{R}$ be the Rees ring of $I$. There is a graded ring $T$ with $\mathcal{R} \subseteq T \subseteq R[u, t]$, such that $T$ is a finite $\mathcal{R}$-module, and* $\operatorname{Ass} T/uT = E(uT)$.

*Proof.* Were we to simply apply Theorem 5 to the ring $\mathcal{R}$ and the regular element $u$, we would find a ring $T$ with $\mathcal{R} \subseteq T \subseteq \mathcal{R}_u = R[u, t]$, such that $T$ is a finite $\mathcal{R}$-module, and $\operatorname{Ass} T/uT = E(uT)$. Thus $T$ would have all the properties we want, except that of being graded. Therefore, this proof shall consist of an outline of what minor changes must be made in the proof of Theorem 5 in order to assure that the resulting $T$ is graded.

Let $S = R - \bigcup \{P \in E(I)\}$. Let $A = R_S$, $B = A^*$, and $C = B/q_1 \oplus \cdots \oplus B/q_n$, where $q_1 \cap \cdots \cap q_n$ is a primary decomposition of 0 in $B$. Let $\mathcal{A} = A[u, IAt]$, $\mathcal{B} = B[u, IBt]$, and $\mathcal{C} = C[u, ICt]$. Now $u$, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ satisfy the hypotheses of Lemma 4. We apply Lemma 3(vi) to $u$ and $\mathcal{C}$, and find a graded ring $H$ with $\mathcal{C} \subseteq H \subseteq C[u, t]$ as described in Lemma 3. We now let $D = H \cap \mathcal{A}_u$. $D$ is as described in Lemma 4, and also, $D$ is a graded ring with $\mathcal{A} \subseteq D \subseteq \mathcal{A}_u = A[u, t]$. Since $\mathcal{R}_S = \mathcal{A}$, and since $S \subseteq R$, we now find $F$ as in the proof of Theorem 5, this time insisting that $F$ is also graded with $\mathcal{R} \subseteq F \subseteq R[u, t]$. Now primes in $E(uF) = Q(uF)$ contract to primes in $E(u\mathcal{R}) = Q(u\mathcal{R})$, and then to primes in $E(I)$. Thus primes in $E(uF)$ are disjoint from $S$. We easily see that $E(uF) = \{P \in \operatorname{Ass} F/uF \mid P$ is contained in some prime in $E(uF)\}$. Finally, apply Lemma 3(vi) to $F$, to find a graded $T$ satisfying our theorem.

DEFINITIONS. If $I$ is an ideal, its integral closure will be denoted $(I)_a$. The ideals $I$ and $J$ are projectively equivalent if for some positive integers $n$ and $m$, $(I^n)_a = (J^m)_a$.

By Proposition 1(g) and (h), we see that $E(I) \subseteq \bigcap A^*(I')$ over all ideals $I'$ which are projectively equivalent to $I$. In [1], it is shown that this inclusion is actually an equality. Our next theorem, goes considerably further, and has this fact as an obvious corollary. Our proof is independent of [1].

THEOREM 9. *Let $I$ be an ideal in a Noetherian ring $R$. Then there is an ideal $J$ projectively equivalent to $I$ such that if $R(J)$ is the Rees ring $R[u, Jt]$, then* $\operatorname{Ass} R(J)/uR(J) = E(uR(J))$. *Furthermore,* $E(I) = A^*(J) = \bigcup \operatorname{Ass} R/J^m$, *over all $m \geqslant 1$. (In fact, for large $n$, we can find $J$ with $I^n \subseteq J \subseteq (I^n)_a$.) Also, there is an ideal $K$ projectively equivalent to $I$ such that $E(I) = \operatorname{Ass} R/K$.*

*Proof.* Let $\mathscr{R} \subseteq T \subseteq R[u, t]$ be as in Theorem 8. Let $I_n = u^n T \cap R$. Suppose $n$ is large enough that a set of homogeneous module generators of $T$ over $\mathscr{R}$ all have degree $n$ or less. Then it is not hard to see that $I_{n+j} = I^j I_n$ for all $j \geqslant 0$, which implies $(I_n)^k = I_{nk}$ for all $k \geqslant 1$. Let $J = I_n$, so that $J^k = I_{nk}$ for all $k \geqslant 1$. Since $T$ is between $\mathscr{R}$ and its integral closure, we see that $I^n \subseteq J \subseteq (I^n)_a$. Let $B = R[u^n, Jt^n] \subseteq T$. Now it is easy to see that $u^n T \cap B = u^n B$. Thus primes in $\operatorname{Ass} B/u^n B$ lift to primes in $\operatorname{Ass} T/u^n T$.

$B \subseteq T$ is an integral extension, since the $n$th power of any homogeneous element of $T$ is in $B$. Also, $T$ is obviously finitely generated (as a ring) over $B$. Thus $T$ is a finite $B$-module. As $t$ is an indeterminate, we easily see that primes in $\operatorname{Ass} T$ contract to primes in $\operatorname{Ass} B$. By Proposition 1(c), primes in $E(u^n T)$ contract to primes in $E(u^n B)$. Combining this fact with the conclusion of the preceding paragraph, and the fact that $\operatorname{Ass} T/u^n T = \operatorname{Ass} T/uT = E(uT) = E(u^n T)$ (the last equality by Proposition 1(g) or 1(d) and the definition), we see that primes in $\operatorname{Ass} B/u^n B$ are in $E(u^n B)$, and so $\operatorname{Ass} B/u^n B = E(u^n B)$. Now $R(J) = R[u, Jt]$ is obviously isomorphic to $R[u^n, Jt^n] = B$, and so the first conclusion of our result is true.

For the second conclusion, by Proposition 1(g) and (h), we see that $E(I) = E(J) \subseteq A^*(J) \subseteq \bigcup \operatorname{Ass} R/J^m$ over $m \geqslant 1$. Now let $P$ be a prime divisor of $J^m$ for some $m \geqslant 1$. As $J^m = u^m R(J) \cap R$, $P$ lifts to a prime divisor $Q$ of $u^m R(J)$. As $u$ is regular, $Q$ is a prime divisor of $uR(J)$. By the first conclusion, already proved, $Q \in E(uR(J))$. By Proposition 1(d) and the definition, $P = Q \cap R \in E(J) = E(I)$. Thus $\bigcup \operatorname{Ass} R/J^m \subseteq E(I)$, which proves the second conclusion.

The final conclusion of the corollary is easy, since we already have $E(I) = A^*(J)$, $J$ as above. For large $k$, $A^*(J) = \operatorname{Ass} R/J^k$, and so we take $K = J^k$.

*Remark.* In the situation of the previous proof, a bit more is true than we have stated. There is a very natural isomorphism between $E(uR(J))$ and $E(uT)$. To see this, of course there is a natural isomorphism between $E(uR(J))$ and $E(uB)$. Now let $P \in E(uB)$, and by Proposition 1(c), let $Q \in E(uT)$ lie over $P$. Since $Q \in \operatorname{Ass} T/uT$, $Q$ is homogeneous. However, the $n$th power of any homogeneous element of $Q$ falls in $P$, so $Q$ is the only prime in $T$ lying over $P$. Thus Proposition 1(c) shows that $E(uB)$ and $E(uT)$ are naturally isomorphic.

We next highlight an interesting analogy between $E(I)$ and $\bar{A}^*(I)$. The definition of $\bar{A}^*(I)$ is given in the appendix. It is known that $\bar{A}^*(I) = \operatorname{Ass} R/(I^n)_a$ for all large $n$. (In fact, in [3], this is used as the definition of $\bar{A}^*(I)$. That the two definitions are equivalent follows fairly easily from [3, Proposition 3.18(i), (ii), (iii), and Proposition 3.19(i) (iii)].)

Corollary 10. *Let I be an ideal in a Noetherian ring R. Then there is an ideal K projectively equivalent to I such that* $E(I) = \text{Ass}\, R/K$, *and* $\bar{A}^*(I) = \text{Ass}\, R/(K)_a$.

*Proof.* By the above comments, we may pick $n$ large enough that $\bar{A}^*(I) = \text{Ass}\, R/(I^m)_a$ for all $m \geqslant n$. By Theorem 9, we then pick $J$ with $I^n \subseteq J \subseteq (I^n)_a$. Let $K = J^k$, as in the proof of Theorem 9. Then $E(I) = \text{Ass}\, R/K$. Also, we easily see that $I^{nk} \subseteq K \subseteq (I^{nk})_a$, so that $(K)_a = (I^{nk})_a$. Thus $\bar{A}^*(I) = \text{Ass}\, R/(I^{nk})_a = \text{Ass}\, R/(K)_a$.

It is of interest to know when all powers of some ideal $I$ are primary. The next corollary is a variation on that theme.

Corollary 11. *Let I be an ideal in a Noetherian ring R. There is an ideal J projectively equivalent to I such that all powers of J are primary to* $P \in \text{Spec}\, R$ *if and only if* $E(I) = \{P\}$.

*Proof.* If $E(I) = \{P\}$, then the $J$ found in Theorem 9 clearly has all of its powers primary to $P$. Conversely, if such a $J$ exists, then $E(I) \subseteq A^*(J) = \{P\}$, so $E(I) = \{P\}$.

Recall that the classical unmixedness theorem states that if $R$ is Cohen–Macaulay, and $I$ is an ideal of the principal class (i.e., $I$ can be generated by $n$ elements, with $n = \text{height}\, I$), then for $m = 1, 2, 3, \ldots$, $\bigcup \text{Ass}\, R/I^m$ consists exactly of the primes minimal over $I$. We present a variation of this.

Corollary 12. *Let R be locally unmixed, and let I be an ideal of the principal class. Then there is an ideal J projectively equivalent to I such that for* $m = 1, 2, 3, \ldots$, $\bigcup \text{Ass}\, R/J^m$ *consists exactly of the primes minimal over J* (*or equivalently, over I*).

*Proof.* By [1, 3.5] and Proposition 1(a), we have that if $R$ is locally unmixed and $I$ is of the principal class, then $E(I)$ consists exactly of the primes minimal over $I$. (Note: In [1], $E(I)$ is denoted $\tilde{A}^*(I)$.) The result now follows from Theorem 9, and the fact that projectively equivalent ideals have the same radical.

In [7, (3.1)], the following result is shown. If $J$ is an ideal in a local ring $R$, and if $R(J)$ is the Rees ring of $J$, then $\{\text{depth}\, P \mid P \in E(uR(J))\} \subseteq \{\text{depth}\, z \mid z \in \text{Ass}\, R^*\}$.

Corollary 13. *Let I be an ideal in a local ring R. There is an ideal J projectively equivalent to I such that* $\{\text{depth}\, P \mid P \in \text{Ass}\, R(J)/uR(J)\} \subseteq \{\text{depth}\, z \mid z \in \text{Ass}\, R^*\}$. *If also R is complete, then equality holds.*

*Proof.* If we pick $J$ as in Theorem 9, then the first part is immediate from [7, (3.1)]. Suppose now that $R$ is complete, and let $z \in \operatorname{Ass} R$. Let $z^* = zR[u, t] \cap R(J)$, so that $z^* \in \operatorname{Ass} R(J)$. Let $p$ be a minimal prime divisor of $(z^*, u)\, R(J)$. Clearly $p \in E(uR(J))$. The proof of [7, (3.1)] shows that depth $p =$ depth $z$.

## APPENDIX

The study of special sets of prime divisors of an ideal in a Noetherian ring has developed fairly rapidly over the last few years, and has suffered from some growing pains. This is particularly true of the notation and terminology. As an example, previous terminology discussed essential primes and $u$-essential primes. However, subsequent progress has revealed that $u$-essential primes are probably the more important of the two, and it is irksome that they had the more ackward name. After much reflection, the authors of this paper have concluded that it will be worth the effort to make some changes. The following table lists them.

| Old | | New | |
|---|---|---|---|
| $A^*(I)$ | unamed | $A^*(I)$ | persistent primes |
| $E(I)$ | essential primes | $Q(I)$ | quintessential primes |
| $U(I)$ | $u$-essential primes | $E(I)$ | essential primes |
| | not previously discussed | $\bar{Q}^*(I)$ | quintasymptotic primes |
| $\bar{A}^*(I)$ | asymptotic primes | $\bar{A}^*(I)$ | asymptotic primes |

*Note.* Since each of these sets is a subset of $A^*(I)$, a prime in any one of these sets is a prime divisor of $I^n$ for all large $n$. Therefore, the word "divisor" can be added to any of these names. Thus "the essential primes of $I$" will be used interchangeably with "the essential prime divisors of $I$."

DEFINITIONS. These definitions refer to the *new* terminology. $A^*(I)$, $Q(I)$, and $E(I)$ are as defined at the start of this paper. $\bar{Q}^*(I) = \{P \in \operatorname{Spec} R \mid I \subseteq P$ and there is a minimal prime $z$ in $R_P^*$ such that $P_P^*$ is minimal over $IR_P^* + z\}$,

$$\bar{A}^*(I) = \{P \cap R \mid P \in \bar{Q}^*(uR(I))\}.$$

*Remarks.* (a) The similarity between the definitions of $Q(I)$ and $\bar{Q}^*(I)$ is obvious, and we hope the new terminology reflects it. (Of course, that similarity induces a similarity between $E(I)$ and $\bar{A}^*(I)$.)

(b) It is known that $\bar{A}^*(I) = \text{Ass } R/(I^n)_a$ for all large $n$, [3, Chap. 3]. The similarity between this characterization, and the definition of $A^*(I)$, justify the similarity between these two symbols.

(c) The overbar in $\bar{A}^*(I)$ and $\bar{Q}^*(I)$ is to emphasize the connection between $\bar{A}^*(I)$ and the integral closures of $I^n$, mentioned in (b), since $(I^n)_a$ is often denoted $\bar{I}^n$.

## REFERENCES

1. D. KATZ, Prime divisors, asymptotic $R$-sequences, and unmixed local rings, *J. Algebra* **95** (1985), 59–71.
2. D. KATZ AND L. J. RATLIFF, JR., $U$-essential prime divisors and sequences over an ideal, *Nagoya Math. J.* **103** (1986), 39–66.
3. S. MCADAM, "Asymptotic Prime Divisors," Lecture Notes in Mathematics, No 1023, Springer-Verlag, New York, 1983.
4. S. MCADAM AND L. J. RATLIFF, JR., Finite Transforms of a Noetherian ring, *J. Algebra* **101** (1986), 479–489.
5. M. NAGATA, On the chain problem of prime ideals, *Nagoya Math. J.* **10** (1956), 51–64.
6. L. J. RATLIFF, JR., Five notes on asymptotic prime divisors, *Math. Z.* **190** (1985), 567–581.
7. L. J. RATLIFF AND D. KATZ, Form rings and equivalence, *Math. Proc. Cambridge Philos. Soc.* **99** (1986), 447–456.